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The role of the symmetry group in the non-uniqueness of a uniform reference. Case study: An isotropic solid body

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Abstract
According to the theory of materially uniform but inhomogeneous bodies, when the symmetry group of a material is continuous it induces a non-uniqueness to the uniform reference. Therefore, it is possible, that by manipulating the symmetry group the inhomogeneity of the material – namely, the dislocations – may cancel out. A solid mathematical framework is constructed in order to describe this situation using the language of exterior calculus. We set down a system of exterior differential equations which when solved render the totality of the uniform references that may be healed by a given symmetry group. From a mathematical point of view these equations have the form of Cartan's equations of structure. We present the generic set of solutions of these equations and then specialize to the particular case of an isotropic solid body.

Keywords
Dislocations, exterior forms, Cartan's equations of structure, exterior differential equations

1. Introduction

The theory of materially uniform but inhomogeneous bodies has been presented by Noll [1] in a seminal paper and expanded by Wang [2]. A recent monograph by Epstein and Elzanowski [3] (see also Esptein [4]) describes this theory in a thorough and apt way while giving variable extensions and applications. Our approach on some topics may be seen in the work of Sfyris [5, 6]. According to this theory, the continuous symmetry group of a material induces a non-uniqueness to the uniform reference. The uniform reference is the quantity that characterizes the field of dislocations in the body.

Thus, it is possible, that by manipulating the symmetry group, the inhomogeneity of the material – namely, the dislocations – may cancel out. The purpose of the present contribution is to establish a rigorous mathematical framework where this situation is described in mathematical terms. Using the theory of exterior calculus [7, 8], we set down a system of exterior differential equations. By solving them we obtain the totality of uniform references that may be healed by a known symmetry group. The system has the form of Cartan's equation of structure.

The generic form of solutions for this system is given. After that we specialize to a solid body. For solid bodies there are two cases where the symmetry group of the material is continuous: isotropy and transverse isotropy [3]. For an isotropic solid body we obtain a system of three recursive forms. We give the generic form of the solution of this system, so we obtain the totality of uniform references that may be healed from the symmetry group of an isotropic solid body.

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The paper is structured as follows. Section 2 gives the statement of the problem in mathematical terms. We start by characterizing the 'healing' of dislocations by the symmetry group of the material and translate it to an exact mathematical framework using the theory of exterior calculus. We parallel this system with Cartan's equations of structure and give the generic set of solutions. Section 3 specializes to the isotropic case. By assuming the symmetry group to consist of orthogonal matrices we solve the system of exterior differential equations, thus finding the totality of uniform references that may be healed by the symmetry group of an isotropic material. Section 4 concludes this approach and directs future developments on the topic.

2. Statement of the problem and general solution

Our body is assumed to be a solid body placed in a reference configuration $\mathcal{B}_{\mathrm{R}}$. The symmetry group of the material is denoted by $\mathcal{C}$ and for an arbitrary element $\mathbf{C} \in \mathcal{C}$ it holds that [9]

$$W(\mathbf{F}) = \tilde{W}(\mathbf{FC}) \tag{1}$$

where $\mathbf{F}$ is the deformation gradient from the tangent space of $\mathcal{B}_{\mathrm{R}}$ to the tangent space of the current configuration $\mathcal{B}_{\mathrm{C}}$. Essentially, $\mathbf{C} \in \mathcal{C}$ is a mapping of the form $T_X\mathcal{B}_{\mathrm{R}} \rightarrow T_X\mathcal{B}_{\mathrm{R}}$ (automorphism) such that equation (1) holds [10], W being the elastic energy.

The field of dislocations is described by the uniform reference $\mathbf{K}^{-1}$ [1, 3]. From the physical point of view it describes the relaxation of the body from the internal stresses due to the dislocations presence [11]. It is a mapping from $\mathcal{B}_{\mathrm{R}}$ to the uniform reference configuration $\mathcal{B}_{\mathrm{U}}$. The latter configuration is obtained by relaxing the body from the internal stresses. When indicial notation is used we write capital Latin indices for quantities in $\mathcal{B}_{\mathrm{R}}$, small Latin indices for quantities in $\mathcal{B}_{\mathrm{C}}$ and small Greek indices for quantities in $\mathcal{B}_{\mathrm{U}}$.

The dislocation density tensor is described by the formula [12, 13]

$$\tilde{\boldsymbol{\alpha}} = \mathrm{Curl}\mathbf{K}^{-1}, \quad \tilde{\alpha}_{C\alpha} = \epsilon_{ABC}K^{-1}_{\alpha A,B} \tag{2}$$

for its two-point expression or by

$$\boldsymbol{\alpha} = \tilde{\boldsymbol{\alpha}}\mathbf{K}^{T}, \quad \alpha_{DC} = \tilde{\alpha}_{D\alpha}K^{T}_{\alpha C} \tag{3}$$

for its full material expression. The permutation symbol is denoted by ϵ.

Given the uniform reference $\mathbf{K}^{-1}$ the symmetry group induces a non-uniqueness described by the formula [1, 3]

$$\hat{\mathbf{K}}^{-1} = \mathbf{K}^{-1}\mathbf{C}, \quad \mathbf{C} \in \mathcal{C}. \tag{4}$$

So, a question that naturally arises is whether or not we can choose $\mathbf{C}$ in such a way that the field $\hat{\mathbf{K}}^{-1}$ corresponds to an ideal elastic material, while the field $\mathbf{K}^{-1}$ corresponds to a truly dislocated body. If this is the case we may say that the dislocations are healed by the symmetry group of the material. This happens when there exists a $\boldsymbol{\varphi} \in C^1(R^3)$ such that

$$\mathbf{K}^{-1}\mathbf{C} = \nabla\boldsymbol{\varphi}, \quad \mathbf{C} \in \mathcal{C}. \tag{5}$$

In this case the symmetry group cancels the field of dislocations out since we have

$$\mathrm{Curl}\mathbf{K}^{-1} \neq \mathbf{0}, \tag{6}$$

while

$$\mathrm{Curl}\hat{\mathbf{K}}^{-1} = \mathbf{0}, \tag{7}$$

$\hat{\mathbf{K}}^{-1}$ being a gradient. In essence, the dislocation density tensor that corresponds to the uniform reference $\hat{\mathbf{K}}^{-1}$ describes a non-dislocated body, while the dislocation density tensor that corresponds to the field $\mathbf{K}^{-1}$ describes a truly defected medium. By applying the Curl operator on both sides of equation (5) we take

$$\mathrm{Curl}\mathbf{K}^{-1}\mathbf{C} + \mathbf{K}^{-1}\mathrm{Curl}\mathbf{C} = 0. \tag{8}$$

We recast the above mathematical framework into the language of exterior calculus. This way we obtain a system of exterior differential equations for the field $\mathbf{K}^{-1}$ when the symmetry group of the material is given. By solving this system we obtain the totality of the uniform references that may be healed by the known symmetry group.

Essentially, we assume that the symmetry group of the material, $\mathbf{C} \in \mathcal{C}$, is given, and we view equation (8) as an equation for $\mathbf{K}^{-1}$ which is the unknown. By solving this equation we will obtain the totality of $\mathbf{K}$s that result in a zero distribution of dislocation density (Curl$\hat{\mathbf{K}}^{-1} = 0$) after the action of the symmetry group according to equation (4) when simultaneously $\mathbf{K}^{-1}$ is Curl-free.

Instead of working with $\mathbf{K}^{-1}$ we utilize its inverse, $\mathbf{K}$. So, equivalently to equation (4) we have

$$\hat{\mathbf{K}} = \mathbf{C}^{-1}\mathbf{K}, \quad \mathbf{C} \in \mathcal{C}. \tag{9}$$

In the language of exterior forms, $\mathbf{K}$ belongs to the space $\Lambda^1_{3,1}$ [8], where $\Lambda^k_{\mathrm{r},s}$ is the space of matrix r by s forms of degree k [7]. So, the field $\mathbf{K}$ is described by three 1-forms. For the components of the symmetry group we have $\mathbf{C} \in \Lambda^0_{3,3}$, hence nine scalar functions, since Λ^0 stands for the space of forms of degree zero, namely scalar functions.

By applying the exterior derivative operator, d, on both sides of equation (9) we obtain [7]

$$\mathrm{d}\hat{\mathbf{K}} = \mathrm{d}\mathbf{C}^{-1} \wedge \mathbf{K} + \mathbf{C}^{-1}\mathrm{d}\mathbf{K}, \tag{10}$$

where the symbol $\wedge$ corresponds to the wedge product following the standard notation of exterior calculus. Since we assume that $\hat{\mathbf{K}}$ corresponds to an ideal elastic body we have

$$\mathrm{d}\hat{\mathbf{K}} = 0, \tag{11}$$

since in this case $\hat{\mathbf{K}}$ corresponds to exact forms. Thus, equation (10) becomes

$$\mathrm{d}\mathbf{K} = -\left(\mathbf{C}\mathrm{d}\mathbf{C}^{-1}\right) \wedge \mathbf{K}. \tag{12}$$

This is the main equation we work with. When the symmetry group is known, $\mathbf{C} \in \mathcal{C}$, equation (12) can be solved for $\mathbf{K}$. This way we obtain the totality of uniform references $\mathbf{K}$ that may be healed from a given symmetry group. Namely, all those $\mathbf{K}$s that after the action of the symmetry group according to equation (9) will give a Curl-free $\hat{\mathbf{K}}$. In order to view equation (10) correctly we have to supplement the necessary integrability conditions. Equations of the form of equation (12) are well studied in the theory of exterior differential systems [7].

We briefly present the fundamental steps in solving general differential systems of exterior forms. Our approach relies heavily to the work of Edelen and Lagoudas [7, 8]. A system of degree k and class r of exterior differential equations on a star-shaped domain is written as

$$\mathrm{d}\Omega = -\Gamma \wedge \Omega + \Sigma, \tag{13}$$
$$\mathrm{d}\Sigma = -\Gamma \wedge \Sigma + \Theta \wedge \Omega, \tag{14}$$
$$\mathrm{d}\Gamma = -\Gamma \wedge \Gamma + \Theta, \tag{15}$$
$$\mathrm{d}\Theta = -\Gamma \wedge \Theta + \Theta \wedge \Gamma, \tag{16}$$

where $\Omega \in \Lambda^k_{r,1}$, $\Gamma \in \Lambda^1_{r,r}$, $\Sigma \in \Lambda^{k+1}_{r,1}$ and $\Theta^2_{r,r}$. When $r = \dim(S)$ and $k = 1$ we refer to Cartan's equations of structure. It should be noted that Γ, Θ and Σ correspond to the connection 1-forms, curvature 2-forms and torsion $(k+1)$-forms of the differential system [equations (13)–16)]. The terminology connection, curvature and torsion is not directly related to the terms that Noll [1] used, for appropriate solderings must done before the juxtaposition of these notions (see Edelen and Lagoudas [8]).

The generic set of solutions for the system in equations (13)–16) is given as [7]

$$\Omega = A\{\mathrm{d}\Phi + \eta - H(\theta \wedge \mathrm{d}\Phi)\}, \tag{17}$$
$$\Sigma = A\{\mathrm{d}\eta + \theta \wedge \eta + H(d\theta \wedge \mathrm{d}\Phi) - \theta \wedge H(\theta \wedge \mathrm{d}\Phi)\}, \tag{18}$$
$$\Gamma = (A\theta - \mathrm{d}A)A^{-1}, \tag{19}$$
$$\Theta = A(\mathrm{d}\theta + \theta \wedge \theta)A^{-1}, \tag{20}$$

where A is the attitude matrix that satisfies the integral equation

$$A = I - H(\Gamma A) \tag{21}$$

and

$$\Phi = H(A^{-1}\Omega), \tag{22}$$
$$\eta = H(A^{-1}\Sigma), \tag{23}$$
$$\theta = H(A^{-1}\Theta A). \tag{24}$$

The linear homotopy operator H is defined by the action on a form of degree k, $\omega = \omega_{i_1 \dots i_k}(x^j)\, \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_k}$ as

$$H\omega = \int_0^1 \mathcal{X} \cdot \tilde{\omega}(\lambda)\lambda^{k-1}\mathrm{d}\lambda, \tag{25}$$

where $\tilde{\omega}(\lambda) = \omega_{i_1 \dots i_k}(x_0^j + \lambda(x^j - x_0^j))\, \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_k}$ is a linear homotopy of ω, $\cdot$ is the inner product between forms and vectors and $\mathcal{X} = (x^i - x_0^i)\partial_i = \{\frac{\partial}{\partial\lambda}(x_0^i + \lambda(x^i - x_0^i))\partial_i\}$ is the radius vector.

When Γ and Σ are assumed to be known then equation (15) is used in order to evaluate Θ. This Θ has to satisfy the conditions of equation (16). What remains in this case is equation (13) which when solved renders all the required Ωs that should satisfy the algebraic constraints of equation (14). Essentially, in this case Ω is the only unknown quantity and the conditions of equations (14)–(16) are integrability conditions.

Now we return to our approach. The starting point is the equation

$$\mathrm{d}\mathbf{K} = -\left(\mathbf{C}\mathrm{d}\mathbf{C}^{-1}\right) \wedge \mathbf{K}. \tag{26}$$

Certainly, it is of the form of equation (13). In place of Ω we have the uniform reference $\mathbf{K}$, and the connection forms are given as

$$\Gamma = \mathbf{C}\mathrm{d}\mathbf{C}^{-1}, \tag{27}$$

while for the torsion forms we have

$$\Sigma = 0. \tag{28}$$

When the symmetry group of the material is given we may evaluate the connection forms.

If we supplement equation (26) with the integrability conditions of the form of equations (14)–(16) we arrive at the following system

$$\mathrm{d}\mathbf{K} = -(\mathbf{C}\mathrm{d}\mathbf{C}^{-1}) \wedge \mathbf{K}, \tag{29}$$
$$\Theta \wedge \mathbf{K} = 0, \tag{30}$$
$$\mathrm{d}\left(\mathbf{C}\mathrm{d}\mathbf{C}^{-1}\right) = -\left(\mathbf{C}\mathrm{d}\mathbf{C}^{-1}\right) \wedge \left(\mathbf{C}\mathrm{d}\mathbf{C}^{-1}\right) + \Theta, \tag{31}$$
$$\mathrm{d}\Theta = -\left(\mathbf{C}\mathrm{d}\mathbf{C}^{-1}\right) \wedge \Theta + \Theta \wedge \left(\mathbf{C}\mathrm{d}\mathbf{C}^{-1}\right). \tag{32}$$

Using equation (31) we may evaluate the curvature form when the connection form is known. This curvature form has to satisfy equation (32) identically. It remains for equation (29) to be solved for $\mathbf{K}$. Finally, these solutions should fulfil the algebraic constraint of equation (30). Thus the generic set of solutions for $\mathbf{K}$ is given by the formula

$$\mathbf{K} = A\{\mathrm{d}\Phi - H(\theta \wedge \mathrm{d}\Phi)\}, \tag{33}$$

where $\eta = 0$, since $\Sigma = 0$ and for the θ and Φ it holds

$$\Phi = H(A^{-1}\mathbf{K}), \tag{34}$$
$$\theta = H(A^{-1}\Theta A), \tag{35}$$

where Θ is found from equation (31) with known connection forms in equation (27).

To sum up, our system is given by equations (29)–(32) and the generic solution for the unknown $\mathbf{K}$ is given by equation (33). So, when $\mathbf{C} \in \mathcal{C}$ is specified we can evaluate the connection forms as $\Gamma = \mathbf{C}\mathrm{d}\mathbf{C}^{-1}$ and from equation (31) we may evaluate the curvature forms that have to also fulfil equation (32). The solution for $\mathbf{K}$ is then of the form of equation (33) and should also satisfy the algebraic constraint of equation (30).

Remarks

i. It is perhaps interesting to note that the framework of Section 2 is valid not only for a solid body, but for a materially uniform but inhomogeneous body of any kind.

ii. The form of our system, since the torsion forms are equal to zero, is essentially a seeking for recursive forms $\mathbf{K} \in \Lambda^1_{3,1}$ [7].

iii. It is perhaps interesting to note that we may recast the main idea of the manuscript using as primary variable the curvature tensor. But, this approach is very difficult to tackle from the mathematical point of view. To see this we recast the problem of determining all those uniform references that may be healed by a known symmetry group in terms of the curvature tensor. The starting point is the relation of the curvature tensor $\mathbf{R}$ with the connection Γ (these two quantities should not be confused with the ones used in the exterior calculus language, for appropriate solderings have to be done before juxtaposing them)

$$R^A_{BCD} = \frac{\partial \Gamma^A_{DB}}{\partial X^C} - \frac{\partial \Gamma^A_{BC}}{\partial X^D} + \Gamma^A_{CE}\Gamma^E_{DB} - \Gamma^A_{DE}\Gamma^E_{CB}.$$

This curvature tensor corresponds to the initial uniform reference $\mathbf{K}^{-1}$. The connection in this case is

$$\Gamma^A_{BC} = \frac{1}{2} G^{-1\ AD} \left(-\frac{\partial G_{BC}}{\partial X^D} + \frac{\partial G_{CD}}{\partial X^B} + \frac{\partial G_{DB}}{\partial X^C} \right),$$

and the metric

$$\mathbf{G} = \mathbf{K}^{-T}\mathbf{K}^{-1}.$$

The action of the symmetry group according to equation (4), namely

$$\hat{\mathbf{K}}^{-1} = \mathbf{K}^{-1}\mathbf{C},$$

$\mathbf{C} \in \mathcal{C}$ being an element of the (continuous) symmetry group, gives a new connection $\hat{\Gamma}$ and a new curvature $\hat{\mathbf{R}}$. Since we assume the dislocations to be healed after the action of the symmetry group the new curvature tensor should be set equal to zero, namely

$$\hat{\mathbf{R}} = \mathbf{0}.$$

The last equation should be viewed as an equation for $\mathbf{K}^{-1}$ when the symmetry group of the material is known. It will play the role of equation (12). The vanishing of the curvature consists of a non-linear equation which is much more difficult to tackle from the mathematical point of view compared to the one arising from adopting the exterior calculus formulation, even though it is an equivalent statement of the problem.

iv. In the above framework nothing is mentioned about the global or local nature of the uniform reference. When the $\mathbf{K}$s have a global character then essentially we refer to the global homogeneity of the body [3]. In this case one may show that the general solution of equation (5) for $\mathbf{K}^{-1}$ consists of all expressions of the form $\mathbf{TC}^{-1}$, where $\mathbf{T}$ is the tangent mapping of a global body diffeomorphism, while $\mathbf{C}$ belongs to the symmetry group. For the case of local homogeneity the uniform reference has a local character thereby providing the possibility for more solutions.

3. Isotropic solid body

In this section we assume our body to be an isotropic solid. We present the solution of the mathematical framework described in Section 2 for the case of an isotropic solid body.

For such a body we know that the symmetry group consists of all rotations for a material point. Thus we have

$$\mathcal{C} = \{\mathbf{Q} \in GL(3; R) : \mathbf{Q}^T = \mathbf{Q}^{-1}\}, \tag{36}$$

so, the symmetry group is the orthogonal group. The general linear group is denoted by $GL(3; R)$. The orthogonal group is a connected and compact Lie group [14], so every element of this group has the representation [8]

$$\mathbf{C} = e^{u^\alpha}\boldsymbol{\gamma}_\alpha, \tag{37}$$

where $\boldsymbol{\gamma}_\alpha$ is a basis for skew matrices while u^α is a system of canonical coordinates on a neighborhood of the identity in the group.

For the basis of the skew matrices we have

$$\boldsymbol{\gamma}_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \boldsymbol{\gamma}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \boldsymbol{\gamma}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (38)$$

Using the representation of equation (37) we may evaluate for $\mathbf{C} = \mathbf{Q}$ [8]

$$\mathrm{d}\mathbf{Q}\mathbf{Q}^{-1} = \lambda_\rho^\beta(u^\alpha)\mathrm{d}u^\rho \boldsymbol{\gamma}_\beta, \quad (39)$$

so using the fact that

$$\mathrm{d}\mathbf{Q}\mathbf{Q}^{-1} = -\mathbf{Q}\mathrm{d}\mathbf{Q}^{-1} \quad (40)$$

we obtain

$$\mathbf{Q}\mathrm{d}\mathbf{Q}^{-1} = -\lambda_\rho^\beta(u^\alpha)\mathrm{d}u^\rho \boldsymbol{\gamma}_\beta. \quad (41)$$

Essentially, the inhomogeneity of the action of the Lie group stems from the inhomogeneity of the quantities u^α. Equation (41) above gives the connection forms for the isotropic solid body problem as follows

$$\Gamma = \mathbf{Q}\mathrm{d}\mathbf{Q}^{-1} = -\lambda_\rho^\beta(u^\alpha)\mathrm{d}u^\rho \boldsymbol{\gamma}_\beta. \quad (42)$$

Using this expression we evaluate for the curvature

$$\Theta = \mathrm{d}\{\mathbf{Q}\mathrm{d}\mathbf{Q}^{-1}\} + \left(\mathbf{Q}\mathrm{d}\mathbf{Q}^{-1}\right) \wedge \left(\mathbf{Q}\mathrm{d}\mathbf{Q}^{-1}\right), \quad (43)$$

the form

$$\Theta = 2\lambda_\rho^\beta(u^\alpha)\lambda_\zeta^\epsilon(u^\delta)\mathrm{d}u^\rho \wedge \mathrm{d}u^\zeta \boldsymbol{\gamma}_\beta \boldsymbol{\gamma}_\epsilon \quad (44)$$

The equation that Θ should satisfy

$$\mathrm{d}\Theta = \left(\mathbf{Q}\mathrm{d}\mathbf{Q}^{-1}\right) \wedge \Theta + \Theta \wedge \left(\mathbf{Q}\mathrm{d}\mathbf{Q}^{-1}\right) \quad (45)$$

results in a set of constraints for λ_ρ^β as follows

$$\frac{\partial \lambda_\rho^\beta}{\partial u^\mu}\lambda_\zeta^\epsilon \boldsymbol{\gamma}_\beta \boldsymbol{\gamma}_\epsilon + \lambda_\rho^\beta \frac{\partial \lambda_\zeta^\epsilon}{\partial u^\mu}\boldsymbol{\gamma}_\beta \boldsymbol{\gamma}_\epsilon + 4\lambda_\rho^\beta \lambda_\zeta^\epsilon \lambda_\mu^\nu \boldsymbol{\gamma}_\beta \boldsymbol{\gamma}_\epsilon \boldsymbol{\gamma}_\nu = 0. \quad (46)$$

What remains in order to find the solution for the uniform reference $\mathbf{K}$ is to evaluate the attitude matrix A. For that purpose we use a standard theorem [7]. According to this theorem, since the connection

$$\Gamma = \mathbf{Q}\mathrm{d}\mathbf{Q}^{-1} \quad (47)$$

take values in the Lie algebra of the orthogonal group, the attitude matrix belongs to the matrix Lie group of rotations for every material point. So, it has the representation

$$A = e^{u^\alpha}\boldsymbol{\gamma}_\alpha. \quad (48)$$

Thus, the solutions for $\mathbf{K}$ that may be healed in an isotropic body are given by the expression

$$\mathbf{K} = e^{u^\alpha}\boldsymbol{\gamma}_\alpha\{\mathrm{d}\Phi - H(\theta \wedge \mathrm{d}\Phi)\}, \quad (49)$$

where

$$\Phi = H(A^{-1}\mathbf{K}), \quad (50)$$

$$\theta = H(A^{-1}\Theta A), \quad (51)$$

with Θ given by equation (44). To sum up, equation (49) gives the totality of all uniform references that may be healed by the orthogonal symmetry group of an isotropic solid body. They should also satisfy the constraint

$$\Theta \wedge \mathbf{K} = 0. \quad (52)$$

4. Conclusions and future work

The purpose of the present contribution is to examine the role played by the symmetry group in the uniqueness of the uniform reference. Using standard notions from exterior calculus, we wrote down a system of exterior differential equations. When the symmetry group is known, solving this system renders the totality of uniform references that may cancel out. The generic solution of this system is given based on standard references on exterior calculus [7]. We specialized to an isotropic solid body and gave the generic expression of the uniform reference that may be healed in such a body.

The non-uniqueness that arises form the inhomogeneous action of the symmetry group calls for a theory where invariant quantities are used. It seems that the gauge covariant derivative utilized in Edelen and Lagoudas [8] has all the required properties for building up invariant – under the inhomogeneous action of the symmetry group – quantities [15].

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Conflict of interest

None declared.

References

[1] Noll, W. Materially uniform simple bodies with inhomogeneities. *Arch Rat Mech Anal* 1967; 27: 1–32.
[2] Wang, C-C. On the geometric structure of simple bodies, a mathematical foundation for the theory of continuous distribution of dislocations. *Arch Rat Mech Anal* 1967; 27: 33–94.
[3] Epstein, M, and Elzanowski, M. *Material inhomogeneities and their evolution. A geometric approach*. New York: Springer, 2007.
[4] Epstein, M. *The geometrical language of continuum mechanics*. Cambridge: Cambridge University Press, 2010.
[5] Sfyris, D, Charalambakis, N, and Kalpakides, VK. Continuously dislocated elastic bodies with a neo Hokean like expression for the energy subjected to antiplane shear. *J Elasticity* 2008; 93: 245–262.
[6] Sfyris, D. Comparing the condition of strong ellipticity and the solvability of an elastic problem and the corresponding dislocated problem. *Math Mech Solids* 2012; 17(3): 254–265 .
[7] Edelen, DGB. *Applied exterior calculus*. Hoboken, NJ: John Wiley & Sons, 1984.
[8] Edelen, DGB, and Lagoudas, D. *Gauge theory and defects in solids*. Amsterdam: North-Holland, 1989.
[9] Marsden, JE, and Hughes, TJR. *Mathematical foundations of elasticity*. Upper Saddle River, NJ: Prentice-Hall, 1983.
[10] Noll, W. A new mathematical theory of simple materials. *Arch Rat Mech Anal* 1972; 48: 1–50.
[11] Epstein, M, and Maugin, GA. The energy momentum tensor and material uniformity in finite elasticity. *Acta Mech* 1990; 83: 127–133.
[12] Kosevich, AM. Crystal dislocations and the theory of elasticity. In: Nabarro, FRN (ed.) *Dislocations in solids. The elastic theory*. Amsterdam: North-Holland, 1979, vol. 1, 33–142.
[13] Kroner, E. Continuum theory of defects. In: Balian, R, and Kleeman, B (eds.) *Physics of defects*. North-Holland: Amsterdam, 1980, 215–315.
[14] Cornwell, JF. *Group theory in physics: an introduction*. New York: Academic Press, 1997.
[15] Sfyris, D. Invariant under the action of the symmetry group quantities in the theory of dislocations, in preparation.